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INFINITUDE AS A PHILOSOPHICAL PROBLEM.

PROFESSOR C. J. KEYSER'S VIEW WITH EDITORIAL COMMENTS.

PROF. CASSIUS JACKSON KEYSER, a mathematician of Columbia University, well versed in the philosophy of mathematics, especially in its more recent development since the appearance of Riemann's famous *Habilitationsschrift*, has published in *The Hibbert Journal* an article on "The Axiom of Infinity," which he criticises in its rôle of "a new presupposition of thought." He says:

"For thousands of years philosophy has recognised the presence of a certain definite problem, namely, that of *extending the dominion of logic, the reign of exact thought, out beyond the utmost reach of finite things into and over the realm of infinite being*, and this problem, by far the greatest and most impressive of her strictly intellectual concerns, philosophy has, for thousands of years, arduously striven to solve. And now I ask—not, has it been worth while? for that is conceded, but—has she advanced the *solution* in any measure, and, if so, in what respect, and to what extent?"

Professor Keyser, continuing, says that "thanks to the subtle genius of the modern Teutonic mind, this ancient problem, having baffled the thought of all the centuries, has at last been completely solved." He then refers to Riemann, Bolzano, Dedekind and Cantor. The first mentioned mathematician distinguishes clearly between boundlessness and infinitude, and Professor Keyser selects for the purpose of his discussion the following definition of "that august term":

"An assemblage (ensemble, collection, group, manifold) of elements (things, no matter what) is infinite or finite according as it

has or has not a PART to which the whole is just EQUIVALENT in the sense that between the elements composing that part and those composing the whole there subsists a unique and reciprocal (one-to-one) correspondence."

For a distinction of the notions of finite and infinite he quotes the mathematical theologian Bolzano, saying:

"Bolzano's procedure is virtually as follows: Suppose given a class *C* of elements, or things, of any kind whatsoever, as the sands of the seashore, or the stars of the firmament, or the points of space, or the instants in a stretch of time, or the numbers with which we count, or the total manifold of truths known to an omniscient God. Out of any such class *C*, suppose a series formed by taking for first term one of the elements of *C*, for second term two of them, and so on. Any term so obtainable is itself obviously a class or group of things, and is *defined* to be finite. The indicated process of series formation, if sufficiently prolonged, will either exhaust *C* or it will not. If it will, *C* is itself *demonstrably* finite; if it will not, *C* is, on that account, *defined* to be infinite. Now, say Professor Royce and others, a definition like the latter, being dependent on such a notion as that of inexhaustibility or endlessness or boundlessness, is negative; a certain innate craving of the understanding remains unsatisfied, we are told, because the definition presents the notion, not in a positive way by telling us what the infinite actually *is*, but merely in a negative fashion by telling us what it is not.

"Undoubtedly the claim is plausible, but is it more? Bolzano affirmed and exemplified a certain proposition, in itself of the utmost importance, and throwing half the needed light upon the question in hand. That proposition is: *Any class or assemblage (of elements), if infinite according to his own definition of the term, enjoys the property of being equivalent, in the sense above explained, to some proper part of itself.* Though he did not himself demonstrate the proposition, it readily admits of demonstration, and, since his time, has in fact been repeatedly and rigorously proved. Not only that, but the converse proposition, giving the other half of the needed light, has been established too: *Every assemblage that HAS a part 'equivalent' to the whole, IS infinite in the Bolzano sense of the term.*

"I turn now to the current assertion by Professor Royce and Mr. Russell, that the modern concept of the infinite, of which I have given above in italics an exact statement, to which the reader is referred, in fact denies a certain ancient axiom of common sense, namely, the axiom of whole and part.

"The question is whether it is possible, by means of the new concept, to demonstrate the existence of the infinite; whether, in other words, it can be proved that there are infinite systems. That such demonstration is possible is affirmed by Bolanzo, by Dedekind, by Professor Royce, by Mr. Russell, and in fact by a large and swelling chorus of authoritative utterance, scarcely relieved by a dissenting voice. After no little pondering of the matter, I have been forced, and that, too, I must own, against my hope and will, to the opposite conviction. Candor, then, compels me to assert, as I have elsewhere briefly done, not only that the arguments which have been actually adduced are all of them vitiated by circularity, but that, in the very nature of conception and inference, by virtue of the most certain standards of logic itself, every potential argument, every possible attempt to prove the proposition, is foredoomed to failure, destined before its birth to take the fatal figure of the wheel.

"The upshot, then, is this: that conception and logical inference alike presuppose absolute certainty that an act which the mind finds itself capable of performing is intrinsically performable endlessly, or, what is the same thing, that the assemblage of possible repetitions of a once mentally performable act is equivalent to some proper part of the assemblage. This certainty I name the *Axiom of Infinity*, and this axiom being, as seen, a necessary presupposition of both conception and deductive inference, every attempt to 'demonstrate' the existence of the infinite is a predestined begging of the issue.

"What follows? Do we, then, *know* by axiom that the infinite is? That depends upon your metaphysic. If you are a radical *a-priorist*, yes; if not, no. If the latter, and I am now speaking as an *a-priorist*, then you are agnostic in the deepest sense, being capable, in utmost rigor of the terms, of neither conceiving nor inferring. But if we do not *know* the axiom to be true, and so cannot deductively prove the existence of the infinite, what, then, is the

probability of such existence? The *highest yet attained*. Why? Because the *inductive* test of the axiom, regarded now as a hypothesis, is trying to conceive and trying to infer, and this experiment, which has been world-wide for æons, has seemed to succeed in countless cases, and to fail in none not explainable on grounds consistent with the retention of the hypothesis.

"Finally, to make briefest application to a single concrete case. Do the stars constitute an infinite multitude? No one knows. If the number be finite, that fact may some time be ascertained by actual enumeration, and, if and only if there be infinite ensembles of possible repetitions of mental processes, it may also be known by proof. But if the multitude of stars be infinite, that can never be known *except* by proof; this last is possible only if the axiom of infinity be true, and even if this be true, the actual proof may never be achieved."

We agree with Professor Keyser when he expresses his conviction that the existence of the infinite cannot be proved, but we venture to supplement this brief statement of his views by the following suggestions: By infinite we understand a process which is to be carried on incessantly. If we think of a mathematical straight line as being produced without limits, we call it infinite. Should we ever try to draw on, even if it were done only in thought, we should soon find out that our line is always of a definite length and never truly infinite, for we would need an infinite time to complete the task. The rigor of logic forces us to admit that infinitude is a process in *action*, but not a concrete and ready thing. Whether the number of stars or the grains of sand on the seashore are infinite or not is a question which can never be decided by experiment, but if our logical laws hold good, and if they possess any value at all, we must admit that (if our existence were widened into a divine omniscience and omnipotence) we could most easily count the grains of sand on our planet (assuming that there is no quibbling about their size as to which ones are mere dust, being too small to be counted) and we could with no less facility determine the number of stars that course within the range of our milky system (provided again that no doubt exists as to which celestial bodies should be regarded as

stars and which as mere meteorites or stellar dust); for anything that is concrete must be definite and anything that is infinite can never be a concrete thing, but must be a process in progress.

This appears to amount to a negation of the existence of infinity; and perhaps it does, at least in the opinion of those metaphysical philosophers who identify the term reality with substantiality, or even with materiality; but the infinite is after all actual, for it inheres in activity which wherever we take it is always an infinite series. Moreover, every particular part of the universe may be considered in its relation to the whole; yet the present moment in its relation to other moments in both the past and the future is but a fleeting point in infinite time, and every spot that determines a definite locality may in all directions and at any imaginal distance be placed in relation to the surrounding world, which renders the proposition obvious that the infinite is the potentiality of actual existence, and is as such not less real than the finite. The present moment alone is truly existent, and the "here" is to us the centre of the universe. It is the place in which our activity is real, but the directions which it can pursue as well as the distance to which, at least in thought, it may venture, are alike infinite.

Infinitude is an evanescent quality; it comes or goes according to the viewpoint we take, according to the task we set ourselves. Take, for instance, the line AB , which may be one mile, or one foot, or one inch long, just as small as you see fit. It is finite; yet you can divide it and there is no limit to your division. It is infinitely divisible. Thus you create infinitude by a conceptual viewpoint. Or take a definite number, e. g. the fraction $\frac{1}{3}$; it is a definite quantity, but if we change it into a decimal fraction, the result is an infinite decimal fraction, viz., $0.3333 \dots$ to be continued without end. If we ever stop the fraction is no longer equal to $\frac{1}{3}$. Yet this infinite decimal exactly equals the unequivocally finite and definite fraction $\frac{1}{3}$. The infinite fraction $0.3333 \dots$ can never be completed, the definite fraction $\frac{1}{3}$ is complete. The infinite and the finite are not two different things, but they are two aspects of the same reality. The finite hangs in the infinite as a temporary and concrete actualisation, and the infinite is inherent in the finite as the

inexhaustible potentiality of its activity. The finite is the changeable, the non-permanent, the transient. The infinite is the resource of all possible existence illimitable in its possibilities.

This conception of infinity seems to be a negation of its existence; and assuredly it is a negation of the notion that infinitude can ever be a concrete thing, realised in any place or at any one time, or in any material body or collection of bodies. But while it denies the materiality, the concreteness, the definiteness (or let us directly add, the finiteness) of infinity, it yet implies its actuality as a most prominent feature of the world-process. Infinity is an intrinsic quality of all activity, and as such it is the most essential part of reality constituting its profundity and the mysterious charm of its eternal youthfulness and freshness; for if it were not, reality would be monotonous, and if not meaningless, certainly both shallow and trite because exhaustible in meaning.